

On Parameter Estimation for Cusp-type Signals

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Abstract

We consider the problem of parameter estimation by the observations of deterministic signal in white gaussian noise. It is supposed that the signal has a singularity of *cusp*-type. The properties of the maximum likelihood and bayesian estimators are described in the asymptotics of small noise. Special attention is paid to the problem of parameter estimation in the situation of misspecification in regularity, i.e.; the statistician supposes that the observed signal has this singularity, but the real signal is smooth. The rate and the asymptotic distribution of the maximum likelihood estimator in this situation are described.

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1 Introduction

Consider the problem of parameter estimation by the observations $X^T = (X_t, 0 \leq t \leq T)$ of the signals in White Gaussian Noise (WGN)

$$dX_t = S(\vartheta, t) dt + \varepsilon dW_t, \quad X_0 = 0, \quad 0 \leq t \leq T. \quad (1)$$

Here $S(\vartheta, t)$ is a known function (signal), $W_t, 0 \leq t \leq T$ is a Wiener process and $\vartheta \in \Theta = (\alpha, \beta)$ is unknown parameter.

We have to estimate the parameter ϑ by continuous time observations X^T and to describe the properties of estimators in the asymptotics of *small noise*, i.e., the parameter $\varepsilon \in (0, 1]$ is known and the asymptotics corresponds to $\varepsilon \rightarrow 0$.

It is known that if the signal $S(\vartheta, \cdot)$ is a smooth function of ϑ , then the maximum likelihood estimator and bayesian estimators are consistent, asymptotically normal

$$\varepsilon^{-1} \left(\hat{\vartheta}_\varepsilon - \vartheta \right) \Longrightarrow \mathcal{N} \left(0, \mathbb{I}(\vartheta)^{-1} \right), \quad \varepsilon^{-1} \left(\tilde{\vartheta}_\varepsilon - \vartheta \right) \Longrightarrow \mathcal{N} \left(0, \mathbb{I}(\vartheta)^{-1} \right),$$

we have the convergence of all polynomial moments and the both estimators are asymptotically efficient [4]. Here $\mathbb{I}(\vartheta)$ is the Fisher information

$$\mathbb{I}(\vartheta) = \int_0^T \dot{S}(\vartheta, t)^2 dt. \quad (2)$$

Here and in the sequel $\dot{}$ means derivation w.r.t. ϑ . If the signal $S(\vartheta, t) = S(t - \vartheta)$, where $S(t)$ is a discontinuous function of t , say, has a jump at the point $t = 0$. Then $\mathbb{I}(\vartheta) = \infty$, the MLE $\hat{\vartheta}_\varepsilon$ and BE $\tilde{\vartheta}_\varepsilon$ have the rate of convergence ε^2 with different limit distributions:

$$\varepsilon^{-2} \left(\hat{\vartheta}_\varepsilon - \vartheta \right) \Longrightarrow \hat{u}, \quad \varepsilon^{-2} \left(\tilde{\vartheta}_\varepsilon - \vartheta \right) \Longrightarrow \tilde{u},$$

and asymptotically efficient are bayesian estimators only. Here $\mathbf{E}(\hat{u})^2 > \mathbf{E}(\tilde{u})^2$. For the proofs see [5].

We are interested by the properties of the MLE $\hat{\vartheta}_\varepsilon$ in the case of observations (1), where the signal $S(\vartheta, t)$ has a singularity of the *cusplike*-type, i.e.; at the vicinity of the point $t = \vartheta$ it has the representation $S(\vartheta, t) \approx a |t - \vartheta|^\kappa$, where $\kappa \in (0, \frac{1}{2})$. Note that for these values of κ we have $\mathbb{I}(\vartheta) = \infty$.

The problem of parameter estimation for cusplike singular density function by i.i.d. observations was considered in [10]. It was shown that the MLE $\hat{\vartheta}_n$ has limit distribution with the rate

$$n^{\frac{1}{2\kappa+1}} \left(\hat{\vartheta}_n - \vartheta \right) \Longrightarrow \hat{\eta}.$$

The exhaustive study of singular estimation problems for i.i.d. observations including cusplike singularity can be found in [6]. For stochastic processes observed in continuous time the similar problems were considered in [2] for inhomogeneous Poisson processes and in [3] for ergodic diffusion processes.

This work is devoted to two problems. The first one is to describe the asymptotics of the MLE and BE in the case of signal with cusp-type singularity. It is shown that

$$\varepsilon^{-\frac{2}{2\kappa+1}} \left(\hat{\vartheta}_\varepsilon - \vartheta \right) \Longrightarrow \hat{\xi}, \quad \varepsilon^{-\frac{2}{2\kappa+1}} \left(\tilde{\vartheta}_\varepsilon - \vartheta \right) \Longrightarrow \tilde{\xi}$$

where $\hat{\xi}$ and $\tilde{\xi}$ are two different r.v.'s, $\mathbf{E}(\hat{\xi}^2) > \mathbf{E}(\tilde{\xi}^2)$. The second problem is to study the properties of the MLE, when the signal supposed by the statistician (theoretical) has cusp-type singularity, but the real signal is smooth (regular). We show that

$$\varepsilon^{-\frac{2}{3-2\kappa}} \left(\hat{\vartheta}_\varepsilon - \hat{\vartheta} \right) \Longrightarrow \hat{\zeta}.$$

Here $\hat{\vartheta}$ is the value of θ which minimizes the corresponding Kulback-Leibler distance. The proofs are carried out following two general results by Ibragimov and Khasminskii (Theorems 1.10.1 and 1.10.2 in [6]), i.e., we verify the conditions of these theorems for our model of observations.

Note that the similar problem of misspecification was considered in the work [1], where the signal chosen by the statistician (theoretical model) has discontinuity, but the real signal is smooth. It is shown that

$$\varepsilon^{-\frac{2}{3}} \left(\hat{\vartheta}_\varepsilon - \hat{\vartheta} \right) \Longrightarrow \hat{\eta}.$$

We discuss as well the problem of estimation κ . The presented work is a continuation of the study [1].

2 Main result

Let us consider the problem of parameter estimation by the observations (in continuous time) of the deterministic signal in the presence of White Gaussian Noise (WGN) of small intensity

$$dX_t = S(\vartheta_0, t) dt + \varepsilon dW_t, \quad X_0 = 0, \quad 0 \leq t \leq T, \quad (3)$$

where the unknown parameter $\vartheta_0 \in \Theta = (\alpha, \beta)$. We are interested by the behavior of the estimators of this parameter in the asymptotics of *small noise*, i.e., as $\varepsilon \rightarrow 0$.

Suppose that the signal $S(\vartheta, t)$ has *cusp*-type singularity

$$S(\vartheta, t) = a |t - \theta|^\kappa + h(t, \vartheta),$$

where $0 < \alpha < \vartheta < \beta < T$ and $\kappa \in (0, \frac{1}{2})$. The function $h(\vartheta, t)$ is continuously differentiable w.r.t. ϑ and has bounded derivative.

The likelihood ratio function is

$$V(\vartheta, X^T) = \exp \left\{ \frac{1}{\varepsilon^2} \int_0^T S(\vartheta, t) dX_t - \frac{1}{2\varepsilon^2} \int_0^T S(\vartheta, t)^2 dt \right\}, \quad \vartheta \in \Theta$$

(see [9]) and the MLE $\hat{\vartheta}_\varepsilon$ is defined by the equation

$$V(\hat{\vartheta}_\varepsilon, X^T) = \sup_{\vartheta \in \Theta} V(\vartheta, X^T). \quad (4)$$

Suppose that ϑ is a random variable with continuous, positive density function $p(\vartheta)$, $\alpha < \vartheta < \beta$. The bayesian estimator (BE) $\tilde{\vartheta}_\varepsilon$ with quadratic loss function is

$$\tilde{\vartheta}_\varepsilon = \frac{\int_\alpha^\beta \theta p(\theta) V(\theta, X^T) d\theta}{\int_\alpha^\beta p(\theta) V(\theta, X^T) d\theta}. \quad (5)$$

We are interested by the properties of the estimators $\hat{\vartheta}_\varepsilon$ and $\tilde{\vartheta}_\varepsilon$ in the asymptotics $\varepsilon \rightarrow 0$.

Note that the Fisher information is not finite and we have a singular problem of parameter estimation. Introduce the Hurst parameter $H = \kappa + \frac{1}{2}$ and double-side fractional Brownian motion (fBm) $W^H(u)$, $u \in R$. Recall, that $\mathbf{E}W_+^H(u) = 0$ and

$$\mathbf{E}W_+^H(u)W_+^H(v) = \frac{1}{2} \left[|u|^{2H} + |v|^{2H} - |u-v|^{2H} \right], \quad u, v \in R. \quad (6)$$

Introduce two random variables $\hat{\xi}$ and $\tilde{\xi}$ by the relations

$$Z(\hat{\xi}) = \sup_{u \in R} Z(u), \quad \tilde{\xi} = \frac{\int u Z(u) du}{\int Z(u) du},$$

where the process

$$Z(u) = \exp \left\{ \Gamma W^H(u) - \frac{\Gamma^2}{2} |u|^{2H} \right\}, \quad u \in R. \quad (7)$$

Here

$$\Gamma^2 = a^2 \int_{-\infty}^{\infty} [|v-1|^\kappa - |v|^\kappa]^2 dv.$$

Introduce as well the process

$$Z^o(v) = \exp \left\{ w^H(v) - \frac{1}{2} |v|^{2H} \right\}, \quad v \in R$$

and the corresponding random variables $\hat{\xi}_o$ and $\tilde{\xi}_o$ by the relations

$$Z(\hat{\xi}_o) = \sup_{v \in R} Z^o(v), \quad \tilde{\xi}_o = \frac{\int v Z^o(v) dv}{\int Z^o(v) dv}.$$

Note that

$$\hat{\xi} = \frac{\hat{\xi}_o}{\Gamma^{\frac{1}{H}}}, \quad \tilde{\xi} = \frac{\tilde{\xi}_o}{\Gamma^{\frac{1}{H}}}. \quad (8)$$

The proof of (8) follows immediately from the change of variables $u = \Gamma^{\frac{1}{H}} v$ in $Z(u)$.

Adymptotically efficient estimators we define with the help of the following lower bound. For all $\vartheta_0 \in \Theta$ and all estimators $\bar{\vartheta}_\varepsilon$ we have the relation

$$\lim_{\delta \rightarrow 0} \underline{\lim}_{\varepsilon \rightarrow 0} \sup_{|\vartheta - \vartheta_0| < \delta} \varepsilon^{-2/H} \mathbf{E}_\vartheta |\bar{\vartheta}_\varepsilon - \vartheta|^2 \geq \mathbf{E}_{\vartheta_0}(\tilde{\xi}^2) = \Gamma^{-\frac{2}{H}} \mathbf{E}(\tilde{\xi}_o^2). \quad (9)$$

Therefore we call the estimator ϑ_ε^* asymptotically efficient if for all $\vartheta_0 \in \Theta$ we have the equality

$$\lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \sup_{|\vartheta - \vartheta_0| < \delta} \varepsilon^{-2/H} \mathbf{E}_\vartheta |\vartheta_\varepsilon^* - \vartheta|^2 = \mathbf{E}_{\vartheta_0}(\tilde{\xi}^2). \quad (10)$$

The proof of this bound follows from the general results presented in [6]. We can recall here the scetch of the proof supposing that the properties of the bayesian estimators for this model are already proved (see theorem 1 below). Introduce a continuous positive density function $(q(\vartheta), \vartheta_0 - \delta < \vartheta < \vartheta_0 + \delta)$. Then we can write

$$\begin{aligned} \sup_{|\vartheta - \vartheta_0| < \delta} \mathbf{E}_\vartheta |\bar{\vartheta}_\varepsilon - \vartheta|^2 &\geq \int_{\vartheta_0 - \delta}^{\vartheta_0 + \delta} \mathbf{E}_\vartheta |\bar{\vartheta}_\varepsilon - \vartheta|^2 q(\vartheta) d\vartheta \\ &\geq \int_{\vartheta_0 - \delta}^{\vartheta_0 + \delta} \mathbf{E}_\vartheta |\tilde{\vartheta}_{q,\varepsilon} - \vartheta|^2 q(\vartheta) d\vartheta. \end{aligned}$$

where we denoted $\tilde{\vartheta}_{q,\varepsilon}$ the bayesian estimator in the case of the density a priory $q(\cdot)$. As we have the convergence of moments of BE we obtain the limit

$$\begin{aligned} \underline{\lim}_{\varepsilon \rightarrow 0} \sup_{|\vartheta - \vartheta_0| < \delta} \varepsilon^{-2/H} \mathbf{E}_\vartheta |\vartheta_\varepsilon^* - \vartheta|^2 &\geq \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2/H} \int_{\vartheta_0 - \delta}^{\vartheta_0 + \delta} \mathbf{E}_\vartheta |\tilde{\vartheta}_{q,\varepsilon} - \vartheta|^2 q(\vartheta) d\vartheta \\ &= \int_{\vartheta_0 - \delta}^{\vartheta_0 + \delta} \mathbf{E}_\vartheta |\tilde{\xi}|^2 q(\vartheta) d\vartheta = \mathbf{E}(\tilde{\xi}^2) = \Gamma^{-\frac{2}{H}} \mathbf{E}(\tilde{\xi}_o^2) \end{aligned}$$

for all $\delta > 0$. Remind that $\mathbf{E}|\tilde{\xi}|^2$ does not depend on ϑ . This proves the lower bound (9).

Theorem 1 *The MLE and BE are consistent, have different limit distributions*

$$\varepsilon^{-\frac{1}{H}} \left(\hat{\vartheta}_\varepsilon - \vartheta \right) \Longrightarrow \hat{\xi}, \quad \varepsilon^{-\frac{1}{H}} \left(\tilde{\vartheta}_\varepsilon - \vartheta \right) \Longrightarrow \tilde{\xi},$$

the polynomial moments converge : for any $p > 0$

$$\lim_{\varepsilon \rightarrow 0} \mathbf{E}_\vartheta \left| \frac{\hat{\vartheta}_\varepsilon - \vartheta}{\varepsilon^{\frac{1}{H}}} \right|^p = \mathbf{E}_\vartheta |\hat{\xi}|^p, \quad \lim_{\varepsilon \rightarrow 0} \mathbf{E}_\vartheta \left| \frac{\tilde{\vartheta}_\varepsilon - \vartheta}{\varepsilon^{\frac{1}{H}}} \right|^p = \mathbf{E}_\vartheta |\tilde{\xi}|^p$$

and the BE are asymptotically efficient.

Proof. To prove this theorem we check the conditions of the general Theorem 1.10.1 in [6]. Let us put $\varphi_\varepsilon = \varepsilon^{1/H}$ and introduce the normalized likelihood ratio

$$Z_\varepsilon(u) = \frac{V(\vartheta_0 + \varphi_\varepsilon u, X^T)}{V(\vartheta_0, X^T)}, \quad u \in \mathbb{U}_\varepsilon = (\varepsilon^{-1/H}(\alpha - \vartheta_0), \varepsilon^{-1/H}(\beta - \vartheta_0)).$$

The verification of these conditions we do with the help of the lemmas presented below.

Lemma 1 *We have the convergence of finite-dimensional distributions of $Z_\varepsilon(\cdot)$: for any set u_1, \dots, u_k and any $k = 1, 2, \dots$*

$$(Z_\varepsilon(u_1), \dots, Z_\varepsilon(u_k)) \Longrightarrow (Z(u_1), \dots, Z(u_k)). \quad (11)$$

This convergence is uniform in ϑ on compacts $\mathbb{K} \subset \Theta$.

Proof. We can write ($u > 0$)

$$\begin{aligned} \ln Z_\varepsilon(u) &= \frac{1}{\varepsilon^2} \int_0^T [S(\vartheta_0 + \varphi_\varepsilon u, t) - S(\vartheta_0, t)] dX_t \\ &\quad - \frac{1}{2\varepsilon^2} \int_0^T [S(\vartheta_0 + \varphi_\varepsilon u, t)^2 - S(\vartheta_0, t)^2] dt \\ &= \frac{1}{\varepsilon} \int_0^T [S(\vartheta_0 + \varphi_\varepsilon u, t) - S(\vartheta_0, t)] dW_t \\ &\quad - \frac{1}{2\varepsilon^2} \int_0^T [S(\vartheta_0 + \varphi_\varepsilon u, t) - S(\vartheta_0, t)]^2 dt. \end{aligned}$$

For the last integral we have

$$\begin{aligned}
& \int_0^T [S(\vartheta_0 + \varphi_\varepsilon u, t) - S(\vartheta_0, t)]^2 dt \\
&= \int_0^T [a|t - \vartheta_0 - \varphi_\varepsilon u|^\kappa - a|t - \vartheta_0|^\kappa + h(\vartheta_0 + \varphi_\varepsilon u, t) - h(\vartheta_0, t)]^2 dt \\
&= \int_{-\vartheta_0}^{T-\vartheta_0} \left[a|t - \varphi_\varepsilon u|^\kappa - a|t|^\kappa + \varphi_\varepsilon u \dot{h}(\tilde{\vartheta}, t - \vartheta_0) \right]^2 dt,
\end{aligned}$$

where we changed the variable and used Taylor expansion for the function $h(\vartheta, t)$.

Let us put $t = \varphi_\varepsilon s$, then we obtain

$$\begin{aligned}
& \int_0^T [S(\vartheta_0 + \varphi_\varepsilon u, t) - S(\vartheta_0, t)]^2 dt \\
&= \varphi_\varepsilon^{2\kappa+1} \int_{-\frac{\vartheta_0}{\varphi_\varepsilon}}^{\frac{T-\vartheta_0}{\varphi_\varepsilon}} \left[a|s - u|^\kappa - a|s|^\kappa + \varphi_\varepsilon^{1-\kappa} u \dot{h}(\tilde{\vartheta}, s\varphi_\varepsilon - \vartheta_0) \right]^2 ds \\
&= a^2 \varphi_\varepsilon^{2\kappa+1} \int_{-\frac{\vartheta_0}{\varphi_\varepsilon}}^{\frac{T-\vartheta_0}{\varphi_\varepsilon}} [|s - u|^\kappa - |s|^\kappa]^2 ds (1 + o(1)).
\end{aligned}$$

Hence

$$\begin{aligned}
& \frac{1}{\varepsilon^2} \int_0^T [S(\vartheta_0 + \varphi_\varepsilon u, t) - S(\vartheta_0, t)]^2 dt \\
&= \frac{a^2 \varphi_\varepsilon^{2\kappa+1}}{\varepsilon^2} \int_{-\frac{\vartheta_0}{\varphi_\varepsilon}}^{\frac{T-\vartheta_0}{\varphi_\varepsilon}} [|s - u|^\kappa - |s|^\kappa]^2 ds (1 + o(1)) \\
&= a^2 |u|^{2\kappa+1} \int_{-\frac{\vartheta_0}{\varphi_\varepsilon u}}^{\frac{T-\vartheta_0}{\varphi_\varepsilon u}} [|v - 1|^\kappa - |v|^\kappa]^2 dv (1 + o(1)) \longrightarrow \Gamma^2 |u|^{2\kappa+1}, \quad (12)
\end{aligned}$$

where we put $s = vu$.

The similar calculations for the stochastic integral provide us the relations

$$\begin{aligned}
& \frac{1}{\varepsilon} \int_0^T [S(\vartheta_0 + \varphi_\varepsilon u, t) - S(\vartheta_0, t)] dW_t \\
&= a \int_{-\frac{\vartheta_0}{\varphi_\varepsilon}}^{\frac{T-\vartheta_0}{\varphi_\varepsilon}} [|s - u|^\kappa - |s|^\kappa] d\tilde{W}(s) (1 + o(1)) \\
&\implies a \int_{-\infty}^{\infty} [|s - u|^\kappa - |s|^\kappa] dW(s) \sim \mathcal{N}\left(0, |u|^{2H} \Gamma^2\right).
\end{aligned}$$

Here $W(v), u \in R$ is two-sided Wiener process

$$W(v) = \begin{cases} W_+(v), & \text{if } v > 0, \\ W_-(-v), & \text{if } v \leq 0, \end{cases}$$

where $W_+(v), W_-(v), v \geq 0$ are two independent Wiener processes.

Let us denote

$$W^H(u) = \Gamma^{-1} \int_{-\infty}^{\infty} [|s - u|^\kappa - |s|^\kappa] dW(s)$$

and verify (6). We use below the equality $ab = \frac{1}{2} [a^2 + b^2 - (a - b)^2]$

$$\begin{aligned} \mathbf{E} W^H(u) W^H(v) &= \frac{1}{2} \left[\mathbf{E} (W^H(u))^2 + \mathbf{E} (W^H(v))^2 - \mathbf{E} (W^H(u) - W^H(v))^2 \right] \\ &= \frac{1}{2} \left[|u|^{2H} + |v|^{2H} - |u - v|^{2H} \right] \end{aligned}$$

because

$$\begin{aligned} \mathbf{E} (W^H(u) - W^H(v))^2 &= \Gamma^{-2} \int_{-\infty}^{\infty} [|s - u|^\kappa - |s - v|^\kappa]^2 ds \\ &= \Gamma^{-2} \int_{-\infty}^{\infty} [|r - (u - v)|^\kappa - |r|^\kappa]^2 dr = |u - v|^{2\kappa+1}. \end{aligned}$$

Hence $W^H(u), u \in R$ is a double-sided fBm.

Therefore we proved the convergence of one-dimensional distributions. The multi-dimensional case is treated by a similar way. We have to verify the convergence

$$\sum_{j=1}^k \lambda_j \ln Z_\varepsilon(u_j) \implies \sum_{j=1}^k \lambda_j \ln Z(u_j).$$

for an arbitrary vectors $(\lambda_1, \dots, \lambda_k)$ and (u_1, \dots, u_k) .

Let us denote

$$\Phi(\vartheta, \vartheta_0) = \int_0^T [S(\vartheta, t) - S(\vartheta_0, t)]^2 dt.$$

We have the following elementary estimate

Lemma 2 *There exists a constant $\mu > 0$ such that*

$$\Phi(\vartheta, \vartheta_0) \geq \mu |\vartheta - \vartheta_0|^{2H}. \quad (13)$$

Proof. Note that for any $\nu > 0$

$$m(\nu) = \inf_{|\vartheta - \vartheta_0| > \nu} \Phi(\vartheta, \vartheta_0) > 0.$$

Indeed, if for some $\nu > 0$ we have $m(\nu) = 0$, then there exists $\vartheta_1 \neq \vartheta_0$ such that for all $t \in [0, T]$

$$a|t - \vartheta_1|^\kappa + h(\vartheta_1, t) = a|t - \vartheta_0|^\kappa + h(\vartheta_0, t)$$

and the function

$$h(\vartheta_1, t) = a|t - \vartheta_0|^\kappa - a|t - \vartheta_1|^\kappa + h(\vartheta_0, t)$$

has no continuous bounded derivativ on ϑ_1 . Hence for $|\vartheta - \vartheta_0| > \nu$

$$\Phi(\vartheta, \vartheta_0) \geq m(\nu) \geq m(\nu) \frac{|\vartheta - \vartheta_0|^{2H}}{|\beta - \alpha|^{2H}}.$$

Further, for the values $|\vartheta - \hat{\vartheta}| \leq \nu$ for sufficiently small ν we have

$$\Phi(\vartheta, \vartheta_0) = |\vartheta - \vartheta_0|^{2H} \Gamma^2(1 + o(1)).$$

Therefore for sufficiently small ν we can write

$$\Phi(\vartheta, \vartheta_0) \geq \frac{1}{2} \Gamma^2 |\vartheta - \vartheta_0|^{2H}.$$

Taking

$$\mu = \min \left(\frac{m(\nu)}{|\beta - \alpha|^{2H}}, \frac{\Gamma^2}{2} \right)$$

we obtain (13).

This estimate allows us to verify the boundness of all moments of the pseudo likelihood ratio process.

Lemma 3 *There exist a constant $c > 0$ such that*

$$\mathbf{E}_{\vartheta_0} Z_\varepsilon^{\frac{1}{2}}(u) \leq e^{-c|u|^{2H}}. \quad (14)$$

Proof. We have

$$\mathbf{E}_{\vartheta_0} Z_\varepsilon(u)^{\frac{1}{2}} = \exp \left\{ -\frac{1}{8\varepsilon^2} \Phi(\vartheta_0 + \varphi_\varepsilon u, \vartheta_0) \right\} \leq \exp \left\{ -\frac{\mu}{8} |u|^{2H} \right\},$$

where we used (13).

Lemma 4 *For any $N > 0$ and $|u_1| < N$, $|u_2| < N$ we have the estimate*

$$\mathbf{E}_{\vartheta_0} \left| Z_\varepsilon^{\frac{1}{2}}(u_2) - Z_\varepsilon^{\frac{1}{2}}(u_1) \right|^2 \leq C(1+N) |u_2 - u_1|^{2H} \quad (15)$$

with some constant $C > 0$.

Proof. We can write

$$\begin{aligned} \mathbf{E}_{\vartheta_0} \left| Z_\varepsilon^{\frac{1}{2}}(u_2) - Z_\varepsilon^{\frac{1}{2}}(u_1) \right|^2 &= 2 \left(1 - \mathbf{E}_{\vartheta_0 + \varphi_\varepsilon u_1} \left(\frac{Z_\varepsilon(u_2)}{Z_\varepsilon(u_1)} \right)^{\frac{1}{2}} \right) \\ &= 2 \left(1 - \exp \left\{ -\frac{1}{8\varepsilon^2} \Phi(\vartheta_0 + \varphi_\varepsilon u_2, \vartheta_0 + \varphi_\varepsilon u_1) \right\} \right) \\ &\leq \frac{1}{4\varepsilon^2} \Phi(\vartheta_0 + \varphi_\varepsilon u_2, \vartheta_0 + \varphi_\varepsilon u_1) \\ &= \frac{1}{4\varepsilon^2} \int_0^T [a|t - \vartheta_0 - \varphi_\varepsilon u_2|^\kappa - a|t - \vartheta_0 - \varphi_\varepsilon u_1|^\kappa \\ &\quad + h(\vartheta_0 + \varphi_\varepsilon u_2, t) - h(\vartheta_0 + \varphi_\varepsilon u_1, t)]^2 dt \\ &\leq \frac{1}{2\varepsilon^2} \int_0^T [a|t - \vartheta_0 - \varphi_\varepsilon u_2|^\kappa - a|t - \vartheta_0 - \varphi_\varepsilon u_1|^\kappa]^2 dt \\ &\quad + \frac{1}{2\varepsilon^2} \int_0^T [h(\vartheta_0 + \varphi_\varepsilon u_2, t) - h(\vartheta_0 + \varphi_\varepsilon u_1, t)]^2 dt \\ &\leq \frac{\varphi_\varepsilon^{2\kappa+1}}{2\varepsilon^2} \Gamma^2 |u_2 - u_1|^{2\kappa+1} + \frac{\varphi_\varepsilon^2}{2\varepsilon^2} \int_0^T \dot{h}(\tilde{\vartheta}, t)^2 dt (u_2 - u_1)^2 \\ &\leq C(1 + |u_2 - u_1|^{1-2\kappa}) |u_2 - u_1|^{2\kappa+1} \leq C(1+N) |u_2 - u_1|^{2H}. \end{aligned}$$

Note that $2\kappa < 1$ and $2H > 1$. The properties of the likelihood ratio (11), (14) and (15) correspond to the conditions of the Theorems 1.10.1 and 1.10.2 in [6] and therefore the MLE $\hat{\vartheta}_\varepsilon$ and BE $\tilde{\vartheta}_\varepsilon$ have all mentioned in the Theorem 1 properties.

Remark 2.1. More detailed analysis shows that if the signal has several points of cusp, say

$$S(\vartheta, t) = \sum_{l=1}^L a_l |t - \vartheta|^{\kappa_l},$$

where $\kappa_l \in (0, \frac{1}{2})$, then the result of the Theorem 1 holds with

$$\kappa = \min_{1 \leq l \leq L} \kappa_l.$$

The proof is similar to the given proof of the Theorem 1.

Remark 2.2. It is possible to study the properties of the estimators $\hat{\vartheta}_\varepsilon$ and $\tilde{\vartheta}_\varepsilon$ in the case of multiple different singularities. For example, suppose that

$$S(\vartheta, t) = \sum_{l=1}^L a_l |t - \vartheta_l|^{\kappa_l},$$

where $\vartheta = (\vartheta_1, \dots, \vartheta_L) \in \Theta$. Here $\Theta = (\alpha_1, \beta_1) \times \dots \times (\alpha_L, \beta_L)$, $0 < \alpha_l < \beta_l < T$ and $\beta_l < \alpha_{l+1}$, $l = 1, \dots, L-1$.

Then the limit for the normalized likelihood ratio

$$Z_\varepsilon(u_1, \dots, u_L) = \frac{V\left(\vartheta_l + \varepsilon^{\frac{1}{H_1}} u_1, \dots, \vartheta_L + \varepsilon^{\frac{1}{H_L}} u_L, X^T\right)}{V(\vartheta_l, \dots, \vartheta_L, X^T)}$$

is the process

$$Z(u_1, \dots, u_L) = \prod_{l=1}^L Z_l(u_l), \quad u_l \in R,$$

where

$$Z_l(u_l) = \exp \left\{ \Gamma_l W_l^{H_l}(u_l) - \frac{\Gamma_l^2}{2} |u_l|^{2H_l} \right\}, \quad u_l \in R,$$

and the constants

$$\Gamma_l^2 = a_l^2 \int_{-\infty}^{\infty} [|v - 1|^{\kappa_l} - |v|^{\kappa_l}]^2 dv.$$

The fBm processes $(W_l^{H_l}(\cdot), \dots, W_L^{H_L}(\cdot))$ are independent. The MLE $\hat{\vartheta}_\varepsilon = (\hat{\vartheta}_{1,\varepsilon}, \dots, \hat{\vartheta}_{L,\varepsilon})$ and BE $\tilde{\vartheta}_\varepsilon = (\tilde{\vartheta}_{1,\varepsilon}, \dots, \tilde{\vartheta}_{L,\varepsilon})$ are defined by the same relations (4), (5) and have different limit distributions. In particular, for the MLE we have the convergence

$$\left(\frac{\hat{\vartheta}_{1,\varepsilon} - \vartheta_1}{\varepsilon^{\frac{1}{H_1}}}, \dots, \frac{\hat{\vartheta}_{L,\varepsilon} - \vartheta_L}{\varepsilon^{\frac{1}{H_L}}} \right) \Rightarrow (\hat{\xi}_1, \dots, \hat{\xi}_L).$$

The limit random variables $(\hat{\xi}_1, \dots, \hat{\xi}_L)$ are defined by the equations

$$Z_l(\hat{\xi}_l) = \sup_u Z_l(u), \quad l = 1, \dots, L$$

and are independent. Of course, the bayesian estimators have the same rate and the asymptotic distribution is

$$\left(\frac{\tilde{\vartheta}_{1,\varepsilon} - \vartheta_1}{\varepsilon^{\frac{1}{H_1}}}, \dots, \frac{\tilde{\vartheta}_{L,\varepsilon} - \vartheta_L}{\varepsilon^{\frac{1}{H_L}}} \right) \Rightarrow (\tilde{\xi}_1, \dots, \tilde{\xi}_L).$$

Here the random variables

$$\tilde{\xi}_l = \frac{\int u_l Z_l(u_l) du_l}{\int Z_l(u_l) du_l}, \quad l = 1, \dots, L$$

are as well asymptotically independent.

3 Misspecification

We are intrerested by the following problem of misspecification. Suppose that the model of observations chosen by the statistician (*theoretical model*) is

$$dX_t = M(\vartheta, t) dt + \varepsilon dW_t, \quad X_0 = 0, \quad 0 \leq t \leq T.$$

The signal $M(\vartheta, t)$ is supposed to be

$$M(\vartheta, t) = a |t - \vartheta|^\kappa, \quad 0 \leq t \leq T,$$

where $\kappa \in (0, \frac{1}{2})$ and $\vartheta \in \Theta = (\alpha < \vartheta < \beta)$. As before we suppose that $0 < \alpha < \beta < T$.

The observed process (*real model*) is

$$dX_t = S(\vartheta_0, t) dt + \varepsilon dW_t, \quad X_0 = 0, \quad 0 \leq t \leq T, \quad (16)$$

where $\vartheta_0 \in \Theta$ is the true value and the function $S(\vartheta, \cdot) \in L_2(0, T)$ is sufficiently smooth.

The likelihood ratio function (misspecified) is

$$V(\vartheta, X^T) = \exp \left\{ \frac{1}{\varepsilon^2} \int_0^T M(\vartheta, t) dX_t - \frac{1}{2\varepsilon^2} \int_0^T M(\vartheta, t)^2 dt \right\}, \quad \vartheta \in \Theta$$

where we have to substitute the observations from the equation (16). Therefore the (pseudo) MLE $\hat{\vartheta}_\varepsilon$ is defined by the equation

$$V(\hat{\vartheta}_\varepsilon, X^T) = \sup_{\vartheta \in \Theta} V(\vartheta, X^T). \quad (17)$$

To see the limit of the MLE we write the likelihood ratio as follows

$$\begin{aligned} \varepsilon^2 \ln V(\vartheta, X^T) &= \varepsilon \int_0^T M(\vartheta, t) dW_t - \frac{1}{2} \int_0^T [M(\vartheta, t)^2 - 2M(\vartheta, t)S(\vartheta_0, t)] dt \\ &= \varepsilon \int_0^T M(\vartheta, t) dW_t - \frac{1}{2} \|M(\vartheta, \cdot) - S(\vartheta_0, \cdot)\|^2 + \frac{1}{2} \|S(\vartheta_0, \cdot)\|^2 \end{aligned}$$

where we denoted as $\|\cdot\|$ the $L_2(0, T)$ norm. It is easy to verify the convergence

$$\sup_{\vartheta \in \Theta} \left| \varepsilon^2 \ln L(\vartheta, X^T) - \frac{1}{2} \|M(\vartheta, \cdot) - S(\vartheta_0, \cdot)\|^2 + \frac{1}{2} \|S(\vartheta_0, \cdot)\|^2 \right| \rightarrow 0.$$

Suppose that the equation

$$\inf_{\vartheta} \|M(\vartheta, \cdot) - S(\vartheta_0, \cdot)\| = \|M(\hat{\vartheta}, \cdot) - S(\vartheta_0, \cdot)\|$$

has a unique solution $\hat{\vartheta} \in \Theta$.

Then we obtain as usual in such situations that the MLE $\hat{\vartheta}_\varepsilon$ converges to the value $\hat{\vartheta}$, which minimizes the Kullback-Leibler distance.

It is interesting to note that in general case $\hat{\vartheta} \neq \vartheta_0$ but sometimes $\hat{\vartheta} = \vartheta_0$ and we consider the conditions of the consistency in such situations. The most interesting for us is the question of the rate of convergence of the MLE to the true value.

Introduce the function

$$\Phi(\vartheta, \hat{\vartheta}) = \|M(\vartheta, \cdot) - S(\vartheta_0, \cdot)\|^2 - \|M(\hat{\vartheta}, \cdot) - S(\vartheta_0, \cdot)\|^2$$

and the conditions of regularity:

Condition \mathcal{M} .

1. The parameter $\kappa \in (0, \frac{1}{2})$.
2. The function $S(\vartheta_0, t)$ for all $\vartheta_0 \in \Theta$ is two times continuously differentiable w.r.t. $t \in [0, T]$.

3. The function $\Phi(\vartheta, \hat{\vartheta})$ for all $\vartheta_0 \in \Theta$ has a unique minimum at the point $\hat{\vartheta} = \hat{\vartheta}(\vartheta_0)$.

4. It's second derivative

$$\gamma(\hat{\vartheta}) \equiv \left. \frac{\partial^2 \Phi(\vartheta, \hat{\vartheta})}{\partial \vartheta^2} \right|_{\vartheta=\hat{\vartheta}} > 0$$

for all $\vartheta_0 \in \Theta$.

Let us denote

$$\begin{aligned} \hat{Z}(u) &= \exp \left\{ aW^H(u) - \frac{\gamma(\hat{\vartheta})}{4} u^2 \right\}, \quad u \in R \\ \hat{Z}^o(u) &= \exp \left\{ w^H(v) - \frac{v^2}{2} \right\}, \quad v \in R \end{aligned}$$

and define the random variables $\hat{\zeta}, \hat{\zeta}_o$ by the relations

$$\hat{Z}(\hat{\zeta}) = \sup_u \hat{Z}(u), \quad \hat{Z}^o(\hat{\zeta}_o) = \sup_v \hat{Z}^o(v).$$

Note that

$$\hat{\zeta} = \left(\frac{2a}{\gamma(\hat{\vartheta})} \right)^{\frac{H}{2H-1}} \hat{\zeta}_o. \quad (18)$$

To verify (18) we change the variables $u = rv$ with $r = (2a)^{\frac{H}{2H-1}} \gamma(\hat{\vartheta})^{-\frac{H}{2H-1}}$ and write

$$\begin{aligned} aW^H(u) - \frac{\gamma(\hat{\vartheta})}{4} u^2 &= aW^H(rv) - \frac{\gamma(\hat{\vartheta})r^2}{4} v^2 \\ &= ar^{\frac{1}{H}} \left(\frac{W^H(rv)}{r^{\frac{1}{H}}} - \frac{\gamma(\hat{\vartheta})r^{2-\frac{1}{H}}}{4a} v^2 \right) = ar^{\frac{1}{H}} \left(w^H(v) - \frac{v^2}{2} \right), \end{aligned}$$

where the fBm $w^H(v) = r^{-\frac{1}{H}} W^H(rv)$.

Theorem 2 Let the conditions \mathcal{M} be fulfilled, then the estimator $\hat{\vartheta}_\varepsilon$ converges to the value $\hat{\vartheta}$, has the limit distribution

$$\frac{\hat{\vartheta}_\varepsilon - \hat{\vartheta}}{\varepsilon^{\frac{2}{3-2\kappa}}} \Longrightarrow \hat{\zeta}, \quad (19)$$

and for any $p > 0$

$$\lim_{\varepsilon \rightarrow 0} \mathbf{E}_\vartheta \left| \frac{\hat{\vartheta}_\varepsilon - \hat{\vartheta}}{\varepsilon^{\frac{2}{3-2\kappa}}} \right|^p = \mathbf{E}_\vartheta |\hat{\zeta}|^p = \left(\frac{2a}{\gamma(\hat{\vartheta})} \right)^{\frac{pH}{2H-1}} \mathbf{E} |\hat{\zeta}_o|^p. \quad (20)$$

Proof. Introduce the normalized pseudo-likelihood ratio process

$$Z_\varepsilon(u) = \frac{V(\hat{\vartheta} + \varphi_\varepsilon u, X^T)}{V(\hat{\vartheta}, X^T)}, \quad u \in \mathbb{U}_\varepsilon = \left(\frac{(\alpha - \hat{\vartheta})}{\varphi_\varepsilon}, \frac{(\beta - \hat{\vartheta})}{\varphi_\varepsilon} \right),$$

where $\varphi_\varepsilon \rightarrow 0$ will be defined later and denote $\vartheta_u = \hat{\vartheta} + \varphi_\varepsilon u$. Below we use the same arguments as that of the preceding section in similar situation ($u > 0$)

$$\begin{aligned} \ln Z_\varepsilon(u) &= \frac{1}{\varepsilon^2} \int_0^T \left[M(\hat{\vartheta} + \varphi_\varepsilon u, t) - M(\hat{\vartheta}, t) \right] dX_t \\ &\quad - \frac{1}{2\varepsilon^2} \int_0^T \left[M(\hat{\vartheta} + \varphi_\varepsilon u, t)^2 - M(\hat{\vartheta}, t)^2 \right] dt \\ &= \frac{1}{\varepsilon} \int_0^T \left[a|t - \vartheta_u|^\kappa - a|t - \hat{\vartheta}|^\kappa \right] dW_t \\ &\quad - \frac{1}{2\varepsilon^2} \int_0^T \left[a|t - \vartheta_u|^\kappa - a|t - \hat{\vartheta}|^\kappa + h(\vartheta_u, t) - h(\hat{\vartheta}, t) \right] \\ &\quad \left[a|t - \vartheta_u|^\kappa + a|t - \hat{\vartheta}|^\kappa - 2S(\vartheta_0, t) \right] dt \\ &= \frac{a\varphi_\varepsilon^{\kappa+\frac{1}{2}}}{\varepsilon} \int_{-\frac{\hat{\vartheta}}{\varphi_\varepsilon}}^{\frac{T-\hat{\vartheta}}{\varphi_\varepsilon}} [|s - u|^\kappa - |s|^\kappa] dW(s) - \frac{1}{2\varepsilon^2} \Phi(\vartheta_u, \hat{\vartheta}). \end{aligned}$$

Let us study the function $\Phi(\vartheta_u, \hat{\vartheta})$ for a fixed $u > 0$ as $\varphi_\varepsilon \rightarrow 0$. We have

$$\begin{aligned} \Phi(\vartheta, \hat{\vartheta}) &= \int_0^T [M(\vartheta, t) - S(\vartheta_0, t)]^2 dt - \int_0^T [M(\hat{\vartheta}, t) - S(\vartheta_0, t)]^2 dt \\ &= \int_{-\vartheta}^{T-\vartheta} [a|s|^\kappa - S(\vartheta_0, s + \vartheta)]^2 dt - \int_0^T [M(\hat{\vartheta}, t) - S(\vartheta_0, t)]^2 dt \end{aligned}$$

and

$$\begin{aligned} \Phi'_\vartheta(\vartheta, \hat{\vartheta}) &= [a|\vartheta|^\kappa - S(\vartheta_0, 0)]^2 - [a|T - \vartheta|^\kappa - S(\vartheta_0, T)]^2 \\ &\quad - 2 \int_{-\vartheta}^{T-\vartheta} [a|s|^\kappa - S(\vartheta_0, s + \vartheta)] S'(\vartheta_0, s + \vartheta) ds. \end{aligned}$$

Recall that as $\hat{\vartheta} \in \Theta$ is the point of minimum of the function $\Phi(\vartheta, \hat{\vartheta})$, $\vartheta \in \Theta$ we have the equalities

$$\Phi(\hat{\vartheta}, \hat{\vartheta}) = 0, \quad \Phi'_\vartheta(\hat{\vartheta}, \hat{\vartheta}) = 0.$$

Let us write the Taylor expansion

$$\begin{aligned}\Phi(\vartheta_u, \hat{\vartheta}) &= \Phi(\hat{\vartheta}, \hat{\vartheta}) + \varphi_\varepsilon u \Phi'_\vartheta(\hat{\vartheta}, \hat{\vartheta}) + \frac{\varphi_\varepsilon^2 u^2}{2} \Phi''_\vartheta(\hat{\vartheta}, \hat{\vartheta}) (1 + o(1)) \\ &= \frac{\varphi_\varepsilon^2 u^2}{2} \Phi''_\vartheta(\hat{\vartheta}, \hat{\vartheta}) (1 + o(1))\end{aligned}$$

and study the difference

$$\begin{aligned}\Phi'_\vartheta(\vartheta_u, \hat{\vartheta}) - \Phi'_\vartheta(\hat{\vartheta}, \hat{\vartheta}) &= [a|\vartheta_u|^\kappa - S(\vartheta_0, 0)]^2 - [a|\hat{\vartheta}|^\kappa - S(\vartheta_0, 0)]^2 \\ &\quad + [a|T - \hat{\vartheta}|^\kappa - S(\vartheta_0, T)]^2 - [a|T - \vartheta_u|^\kappa - S(\vartheta_0, T)]^2 \\ &\quad - 2 \int_{-\vartheta_u}^{T-\vartheta_u} [a|s|^\kappa - S(\vartheta_0, s + \vartheta_u)] S'(\vartheta_0, s + \vartheta_u) ds \\ &\quad + 2 \int_{-\hat{\vartheta}}^{T-\hat{\vartheta}} [a|s|^\kappa - S(\vartheta_0, s + \hat{\vartheta})] S'(\vartheta_0, s + \hat{\vartheta}) ds.\end{aligned}$$

We have the estimates

$$\begin{aligned}[a|\vartheta_u|^\kappa - S(\vartheta_0, 0)]^2 - [a|\hat{\vartheta}|^\kappa - S(\vartheta_0, 0)]^2 &= a \left[|\hat{\vartheta} + \varphi_\varepsilon u|^\kappa - |\hat{\vartheta}|^\kappa \right] \left[a|\hat{\vartheta} + \varphi_\varepsilon u|^\kappa + a|\hat{\vartheta}|^\kappa - 2S(\vartheta_0, 0) \right] \\ &= a \left[|\hat{\vartheta} + \varphi_\varepsilon u|^\kappa - |\hat{\vartheta}|^\kappa \right] \left[a|\hat{\vartheta} + \varphi_\varepsilon u|^\kappa + a|\hat{\vartheta}|^\kappa - 2S(\vartheta_0, 0) \right] \\ &= \frac{2a\kappa}{\hat{\vartheta}^{1-\kappa}} \left[a|\hat{\vartheta}|^\kappa - S(\vartheta_0, 0) \right] \varphi_\varepsilon u + O(\varphi_\varepsilon^2 u^2)\end{aligned}$$

and similiary

$$\begin{aligned}[a|T - \hat{\vartheta}|^\kappa - S(\vartheta_0, T)]^2 - [a|T - \vartheta_u|^\kappa - S(\vartheta_0, T)]^2 &= \frac{2a\kappa}{|T - \hat{\vartheta}|^{1-\kappa}} \left[a|T - \hat{\vartheta}|^\kappa - S(\vartheta_0, T) \right] \varphi_\varepsilon u + O(\varphi_\varepsilon^2 u^2)\end{aligned}$$

because

$$\begin{aligned}|\hat{\vartheta} + \varphi_\varepsilon u|^\kappa - |\hat{\vartheta}|^\kappa &= |\hat{\vartheta}|^\kappa \left(1 + \frac{\kappa \varphi_\varepsilon u}{\hat{\vartheta}} \right) - |\hat{\vartheta}|^\kappa + O(\varphi_\varepsilon^2 u^2) \\ &= \frac{\kappa \varphi_\varepsilon u}{\hat{\vartheta}^{1-\kappa}} + O(\varphi_\varepsilon^2 u^2).\end{aligned}$$

Further, we can write

$$\begin{aligned}
& \int_{-\hat{\vartheta}}^{T-\hat{\vartheta}} \left[a |s|^\kappa - S(\vartheta_0, s + \hat{\vartheta}) \right] S'(\vartheta_0, s + \hat{\vartheta}) \, ds \\
& - 2 \int_{-\vartheta_u}^{T-\vartheta_u} [a |s|^\kappa - S(\vartheta_0, s + \vartheta_u)] S'(\vartheta_0, s + \vartheta_u) \, ds \\
& = \int_0^T \left| t - \hat{\vartheta} \right|^\kappa [S'(\vartheta_0, t + \varphi_\varepsilon u) - S'(\vartheta_0, t)] \, dt \\
& + \int_0^T [S(\vartheta_0, t + \varphi_\varepsilon u) S'(t + \varphi_\varepsilon u) - S(\vartheta_0, t) S'(\vartheta_0, t)] \, dt \\
& + \left(\int_{-\vartheta_u}^{-\hat{\vartheta}} - \int_{T-\vartheta_u}^{T-\hat{\vartheta}} \right) [a |s|^\kappa - S(\vartheta_0, s + \vartheta_u)] S'(\vartheta_0, s + \vartheta_u) \, ds.
\end{aligned}$$

Therefore we obtain the relations

$$\begin{aligned}
& \int_0^T \left| t - \hat{\vartheta} \right|^\kappa [S'(\vartheta_0, t + \varphi_\varepsilon u) - S'(\vartheta_0, t)] \, dt \\
& = \int_0^T \left| t - \hat{\vartheta} \right|^\kappa S''(\vartheta_0, t) \, dt \, \varphi_\varepsilon u + O(\varphi_\varepsilon^2 u^2), \\
& \int_0^T [S(\vartheta_0, t + \varphi_\varepsilon u) S'(\vartheta_0, t + \varphi_\varepsilon u) - S(\vartheta_0, t) S'(\vartheta_0, t)] \, dt \\
& = \frac{1}{2} \int_0^T [S(\vartheta_0, t)^2]_t'' \, dt \, \varphi_\varepsilon u + O(\varphi_\varepsilon^2 u^2), \\
& \int_{-\vartheta_u}^{-\hat{\vartheta}} [a |s|^\kappa - S(\vartheta_0, s + \vartheta_u)] S'(\vartheta_0, s + \vartheta_u) \, ds \\
& = \left[a \left| \hat{\vartheta} \right|^\kappa - S(\vartheta_0, 0) \right] S'(\vartheta_0, 0) \, \varphi_\varepsilon u + O(\varphi_\varepsilon^2 u^2), \\
& \int_{T-\vartheta_u}^{T-\hat{\vartheta}} [a |s|^\kappa - S(\vartheta_0, s + \vartheta_u)] S'(\vartheta_0, s + \vartheta_u) \, ds \\
& = \left[a \left| T - \hat{\vartheta} \right|^\kappa - S(\vartheta_0, T) \right] S'(\vartheta_0, T) \, \varphi_\varepsilon u + O(\varphi_\varepsilon^2 u^2).
\end{aligned}$$

All these together allows us to write

$$\begin{aligned}
\frac{\Phi'_\vartheta(\vartheta_u, \hat{\vartheta})}{\varphi_\varepsilon u} &= \frac{2a\kappa}{\hat{\vartheta}^{1-\kappa}} \left[a |\hat{\vartheta}|^\kappa - S(\vartheta_0, 0) \right] + \frac{2a\kappa}{|T - \hat{\vartheta}|^{1-\kappa}} \left[a |T - \hat{\vartheta}|^\kappa - S(\vartheta_0, T) \right] \\
&+ \left[a \left| \hat{\vartheta} \right|^\kappa - S(\vartheta_0, 0) \right] S'(\vartheta_0, 0) + \left[a \left| T - \hat{\vartheta} \right|^\kappa - S(\vartheta_0, T) \right] S'(\vartheta_0, T) \\
&+ 2 \int_0^T \left| t - \hat{\vartheta} \right|^\kappa S''(\vartheta_0, t) \, dt + \int_0^T [S(\vartheta_0, t)^2]_t'' \, dt + O(\varphi_\varepsilon u). \quad (21)
\end{aligned}$$

Hence we obtain the following expression for second derivative

$$\begin{aligned}
\Phi''_{\vartheta}(\hat{\vartheta}, \hat{\vartheta}) &= \lim_{\varphi_{\varepsilon} \rightarrow 0} \frac{\Phi'_{\vartheta}(\vartheta_u, \hat{\vartheta}) - \Phi'_{\vartheta}(\hat{\vartheta}, \hat{\vartheta})}{\varphi_{\varepsilon} u} \\
&= \frac{2a\kappa}{\hat{\vartheta}^{1-\kappa}} \left[a|\hat{\vartheta}|^{\kappa} - S(\vartheta_0, 0) \right] + \frac{2a\kappa}{|T - \hat{\vartheta}|^{1-\kappa}} \left[a|T - \hat{\vartheta}|^{\kappa} - S(\vartheta_0, T) \right] \\
&\quad + \left[a|\hat{\vartheta}|^{\kappa} - S(\vartheta_0, 0) \right] S'(\vartheta_0, 0) + \left[a|T - \hat{\vartheta}|^{\kappa} - S(\vartheta_0, T) \right] S'(\vartheta_0, T) \\
&\quad + 2 \int_0^T |t - \hat{\vartheta}|^{\kappa} S''(\vartheta_0, t) dt + \int_0^T [S(\vartheta_0, t)^2]_t'' dt. \tag{22}
\end{aligned}$$

Now the log-likelihood ratio has the representation

$$\begin{aligned}
\ln Z_{\varepsilon}(u) &= \frac{a\varphi_{\varepsilon}^{\kappa+\frac{1}{2}}}{\varepsilon} W^H(u) (1 + o(1)) - \frac{\varphi_{\varepsilon}^2 u^2}{4\varepsilon^2} \Phi''_{\vartheta}(\hat{\vartheta}, \hat{\vartheta}) (1 + o(1)) \\
&= \frac{\varphi_{\varepsilon}^{\kappa+\frac{1}{2}}}{\varepsilon} \left(aW^H(u) (1 + o(1)) - \frac{\varphi_{\varepsilon}^{\frac{3}{2}-\kappa}}{\varepsilon} \Phi''_{\vartheta}(\hat{\vartheta}, \hat{\vartheta}) \frac{u^2}{4} (1 + o(1)) \right).
\end{aligned}$$

Therefore if we put

$$\frac{\varphi_{\varepsilon}^{\frac{3}{2}-\kappa}}{\varepsilon} = 1, \quad \varphi_{\varepsilon} = \varepsilon^{\frac{2}{3-2\kappa}}, \quad \hat{Z}_{\varepsilon}(u) = Z_{\varepsilon}(u)^{\varepsilon^{\frac{4\kappa-2}{3-2\kappa}}},$$

then we obtain the convergence of finite-dimensional distributions

$$\left(\hat{Z}_{\varepsilon}(u_1), \dots, \hat{Z}_{\varepsilon}(u_k) \right) \Longrightarrow \left(\hat{Z}(u_1), \dots, \hat{Z}(u_k) \right)$$

for any $k = 1, 2, \dots$

Using the same arguments as in the proofs of the lemmata 2-4 we obtain the relations

$$\begin{aligned}
\Phi(\vartheta, \hat{\vartheta}) &\geq \mu (\vartheta - \hat{\vartheta})^2, \\
\mathbf{E}_{\vartheta_0} \hat{Z}_{\varepsilon}^{\frac{1}{2}}(u) &\leq e^{-cu^2}, \\
\mathbf{E}_{\vartheta_0} \left[\hat{Z}_{\varepsilon}^{\frac{1}{2}}(u_2) - \hat{Z}_{\varepsilon}^{\frac{1}{2}}(u_1) \right]^2 &\leq C(1+N) |u_2 - u_1|^2
\end{aligned}$$

Therefore once more the asymptotic properties of the pseudo-MLE $\hat{\vartheta}_{\varepsilon}$ follow from the general result by Ibragimov and Kasminskii [6], Theorem 1.10.1.

Let us remind how the properties of $\hat{\vartheta}_\varepsilon$ are related with the convergence of the stochastic processes $\hat{Z}_\varepsilon(\cdot) \Rightarrow \hat{Z}(\cdot)$: we can write

$$\begin{aligned}
\mathbf{P}_{\vartheta_0} \left(\frac{\hat{\vartheta}_\varepsilon - \hat{\vartheta}}{\varphi_\varepsilon} < x \right) &= \mathbf{P}_{\vartheta_0} \left(\hat{\vartheta}_\varepsilon < \hat{\vartheta} + \varphi_\varepsilon x \right) \\
&= \mathbf{P}_{\vartheta_0} \left\{ \sup_{\vartheta < \hat{\vartheta} + \varphi_\varepsilon x} V(\vartheta, X^T) > \sup_{\vartheta \geq \hat{\vartheta} + \varphi_\varepsilon x} V(\vartheta, X^T) \right\} \\
&= \mathbf{P}_{\vartheta_0} \left\{ \sup_{\vartheta < \hat{\vartheta} + \varphi_\varepsilon x} \frac{V(\vartheta, X^T)}{V(\hat{\vartheta}, X^T)} > \sup_{\vartheta \geq \hat{\vartheta} + \varphi_\varepsilon x} \frac{V(\vartheta, X^T)}{V(\hat{\vartheta}, X^T)} \right\} \\
&= \mathbf{P}_{\vartheta_0} \left\{ \sup_{u < x, u \in \mathbb{U}_\varepsilon} Z_\varepsilon(u) > \sup_{u \geq x, u \in \mathbb{U}_\varepsilon} Z_\varepsilon(u) \right\} \\
&= \mathbf{P}_{\vartheta_0} \left\{ \sup_{u < x, u \in \mathbb{U}_\varepsilon} \hat{Z}_\varepsilon(u) > \sup_{u \geq x, u \in \mathbb{U}_\varepsilon} \hat{Z}_\varepsilon(u) \right\} = \mathbf{P}_{\vartheta_0}(\hat{u}_\varepsilon < x), \quad (23)
\end{aligned}$$

where $\hat{u}_\varepsilon = \frac{\hat{\vartheta}_\varepsilon - \hat{\vartheta}}{\varphi_\varepsilon}$ is defined by the relation

$$\hat{Z}_\varepsilon(\hat{u}_\varepsilon) = \sup_{u \in \mathbb{U}_\varepsilon} \hat{Z}_\varepsilon(u).$$

Now from the convergence $\hat{Z}_\varepsilon(\cdot) \Rightarrow \hat{Z}(\cdot)$ we obtain

$$\begin{aligned}
&\mathbf{P}_{\vartheta_0} \left\{ \sup_{u < x, u \in \mathbb{U}_\varepsilon} \hat{Z}_\varepsilon(u) > \sup_{u \geq x, u \in \mathbb{U}_\varepsilon} \hat{Z}_\varepsilon(u) \right\} \\
&\longrightarrow \mathbf{P}_{\vartheta_0} \left\{ \sup_{u < x} \hat{Z}(u) > \sup_{u \geq x} \hat{Z}(u) \right\} = \mathbf{P}_{\vartheta_0}(\hat{\zeta} < x)
\end{aligned}$$

(see the details in [6], Theorem 1.10.1).

Remark 3.1. Of course, it is possible to consider slightly more general problem with the signal

$$S(\vartheta, t) = a|t - \vartheta|^\kappa \mathbb{I}_{\{t < \vartheta\}} + b|t - \vartheta|^\kappa \mathbb{I}_{\{t \geq \vartheta\}} + h(\vartheta, t),$$

where $b > 0$ and $h(\vartheta, t)$ is some smooth function of ϑ and t . As usual in singular estimation problems the limit likelihood ratio $Z(\cdot)$ does not depend on the function $h(\cdot, \cdot)$ and the properties of the pseudo-MLE are quite close to that of the presented in the Theorem 2.

There are another interesting problems of misspecification *cusp vs discontinuous* and *discontinuous vs cusp*, which can be illustrated by the following

example. Suppose that we have two signals

$$S(\vartheta, t) = -a |t - \vartheta|^\kappa \mathbb{I}_{\{t < \vartheta\}} + a |t - \vartheta|^\kappa \mathbb{I}_{\{t \geq \vartheta\}},$$

where $\kappa \in (0, \frac{1}{2})$ and

$$M(\vartheta, t) = a \operatorname{sgn}(t - \vartheta).$$

One problem is the estimation of the parameter ϑ in the situation, where $S(\vartheta_0, t)$ is the observed signal and $M(\vartheta, t)$ is supposed (theoretical) signal. The second problem corresponds to the situation where the observed signal is $M(\vartheta_0, t)$ and the theoretical signal is $S(\vartheta, t)$. The both problems are studied in the forthcoming paper.

4 Estimation of the parameter κ

Let us consider the problem of estimation of the parameter $\kappa \in (k, K)$, $0 < k < K < \infty$ by observations

$$dX_t = a |t - \rho|^{\kappa_0} dt + \varepsilon dW_t, \quad X_0 = 0, \quad 0 \leq t \leq T,$$

where $a > 0$ and $\rho \in (0, T)$ are some known parameters. The likelihood-ratio function is

$$V(\kappa, X^T) = \exp \left\{ \int_0^T \frac{a |t - \rho|^\kappa}{\varepsilon^2} dX_t - \int_0^T \frac{a^2 |t - \rho|^{2\kappa}}{2\varepsilon^2} dt \right\}, \quad \kappa \in (k, K)$$

and the MLE $\hat{\kappa}_\varepsilon$ is solution of the equation

$$V(\hat{\kappa}_\varepsilon, X^T) = \sup_{\kappa \in (k, K)} V(\kappa, X^T)$$

This is regular problem with the Fisher information

$$\mathbb{I}(\kappa) = a^2 \int_0^T |t - \rho|^{2\kappa} (\ln |t - \rho|)^2 dt > 0.$$

It is easy to see that the identification condition

$$\inf_{|\kappa - \kappa_0| > \nu} \int_0^T (|t - \rho|^{2\kappa} - |t - \rho|^{2\kappa_0})^2 dt > 0.$$

is fulfilled for any κ_0 and any $\nu > 0$.

Therefore the asymptotic normality

$$\frac{\hat{\kappa}_\varepsilon - \kappa_0}{\varepsilon} \Longrightarrow \mathcal{N}(0, \mathbb{I}(\kappa_0)^{-1})$$

follows from the general theorem devoted to the parameter estimation in regular families (see Theorem 3.1.1 in [6]). Just note that the normalized likelihood ratio

$$Z_\varepsilon^*(v) = \frac{V(\kappa_0 + \varepsilon v, X^T)}{V(\kappa_0, X^T)}, \quad v \in \mathbb{V}_\varepsilon = \left(\frac{k - \kappa_0}{\varepsilon}, \frac{K - \kappa_0}{\varepsilon} \right)$$

converges to the process

$$Z^*(v) = \exp \left\{ v\Delta - \frac{u^2}{2} \mathbb{I}(\kappa_0) \right\}, \quad v \in R, \quad (24)$$

where $\Delta \sim \mathcal{N}(0, \mathbb{I}(\kappa_0))$

It is interesting as well to consider the problem of two-dimensional parameter $\vartheta = (\rho, \kappa)$ estimation. The likelihood-ratio function is

$$V(\rho, \kappa, X^T) = \exp \left\{ \int_0^T \frac{a|t - \rho|^\kappa}{\varepsilon^2} dX_t - \int_0^T \frac{a^2|t - \rho|^{2\kappa}}{2\varepsilon^2} dt \right\}, \quad \vartheta \in \Theta,$$

where $\Theta = (\alpha, \beta) \times (k, K)$, $0 < \alpha < \beta < T$.

It can be shown that the normalized likelihood ratio

$$Z_\varepsilon(u, v) = \frac{V(\rho_0 + \varepsilon^{\frac{1}{H}}u, \kappa_0 + \varepsilon v, X^T)}{V(\rho_0, \kappa_0, X^T)}$$

converges to the random process

$$Z(u, v) = Z(u) Z^*(v)$$

where the processes $Z(\cdot)$ and $Z^*(\cdot)$ are defined by the expressions (7) and (24). Note that the fBm $W^H(\cdot)$ and the random variable Δ are independent.

The MLE $\hat{\vartheta}_\varepsilon = (\hat{\rho}_\varepsilon, \hat{\kappa}_\varepsilon)$ is consistent and its components $\hat{\rho}_\varepsilon$ and $\hat{\kappa}_\varepsilon$ are asymptotically independent and have limit distributions with different normalizing rates

$$\frac{\hat{\rho}_\varepsilon - \rho_0}{\varepsilon^{\frac{1}{H}}} \Longrightarrow \hat{\xi}, \quad \frac{\hat{\kappa}_\varepsilon - \kappa_0}{\varepsilon} \Longrightarrow \frac{\Delta}{\mathbb{I}(\kappa_0)} \sim \mathcal{N}(0, \mathbb{I}(\kappa_0)^{-1}).$$

The proof follows the main steps of the proof of the Theorem 1 is cumbersome and do not presented here.

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